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ON THE PROBABILITY OF INITIATING A PERSISTENT FISSION CHAIN

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ABSTRACT

We derive and discuss the integro-differential equation for the probability of initiating a persistent fission chain. The probability is considered to be a function of space, energy, and time. For the time-independent case, we describe the SNP code, which is a form of the DSN code for efficiently solving the nonlinear probability problem. Additional nuclear parameters which are required for this probability calculation are noted, and the results of a few test problems are shown.

INTRODUCTION

In his classic paper on the statistical behavior of neutron chains,[1] Feynman considered the space and energy-dependent probability of obtaining, from a single neutron, a divergent neutron chain in a stationary system. He showed that this probability satisfies a nonlinear integral equation, the kernel of which is the same as that for the linear neutron transport problem. Numerical solutions of this integral equation were obtained for metal spheres of ... with one neutron velocity group by Blocker[2] in 1953. When Carlson devised his S_n method for numerical solution of the transport equation, it was seen by Lee, Longmire, Goad, and others that this method should be applicable to the solution of the nonlinear probability problem as well, and a one velocity code was devised for this purpose in 1955 by Goad.

In this paper, we first derive and discuss the integro-differential equation for the probability of a divergent chain, which is considered to be a function of space, energy, and time. For the time-independent case,

we then describe the SNP code which is a form of the DSN code for efficiently solving the nonlinear probability problem. The additional nuclear parameters required for the probability problem are noted, and the results of a few test problems are shown.

FUNDAMENTAL EQUATIONS

We now derive an integro-differential equation for the probability of obtaining a divergent chain. The notation is as follows:

Let $p(\vec{r},\vec{\Omega},v,t)$ be the probability that a neutron at position $\vec{r}$, time t, with direction $\vec{\Omega}$, and speed v produces a divergent chain reaction.

Let $q(\vec{r},\vec{\Omega},v,t)$ be the probability that such a neutron does not produce a divergent chain so that $q = 1 - p$.

Let $\sigma(\vec{r},\vec{\Omega},v,t)$ be the probability per cm that the neutron will have a collision.

Let $c_i(\vec{r},\vec{\Omega},v,t \rightarrow \vec{\Omega}'_1 v'_1, \ldots, \vec{\Omega}'_i v'_i)$ be the probability that i neutrons emerge from the collision having velocities $v_1\vec{\Omega}_1$, $v_2\vec{\Omega}_2$ up to $v_i\vec{\Omega}_i$. It is assumed that the neutrons emerge from the collision instantaneously.

Let $\vec{s}$ be a vector starting at $\vec{r}$ and in the direction $\vec{\Omega}$, $\vec{s} = s\vec{\Omega}$, and let the intersection of $\vec{s}$ and the fixed boundary of the problem be $\vec{r}_s$. It is assumed that no material lies beyond $\vec{r}_s$.

We can now write an integral equation which states that the probability of no divergent chain, q, equals the probability that the neutron escapes from the system without collision plus the probability that it collides somewhere, and the resulting neutrons lead to no divergent chains. Thus (with $\vec{r}' = \vec{r} + s\vec{\Omega}$)

$$q(\vec{r},\vec{\Omega},v,t) = \exp\left\{-\int_{\vec{r}}^{\vec{r}_s}\sigma(\vec{r}',\vec{\Omega},v,t+\frac{s}{v})d\vec{r}'\right\}$$

$$+\int_{\vec{r}}^{\vec{r}_s}\sigma(\vec{r}',\vec{\Omega},v,t+\frac{s}{v})\exp\left\{-\int_{\vec{r}}^{\vec{r}'}\sigma(\vec{r}'',\vec{\Omega},v,t+\frac{s'}{v})d\vec{r}''\right\}$$

$$\cdot\, d\vec{r}'\Big[c_0(\vec{r}',\vec{\Omega},v,t+\frac{s}{v})+\int\int dv'd\vec{\Omega}'c_1(\vec{r}',\vec{\Omega},v,t+\frac{s}{v}\to\vec{\Omega}',v')q(\vec{r}',\vec{\Omega}',v,t+\frac{s}{v})$$

$$+\cdots+\int\int\cdots\int\int c_i(\vec{r}',\vec{\Omega},v,t+\frac{s}{v}\to\vec{\Omega}_1',v_1',\ \cdots,\ \vec{\Omega}_i',v_i')$$

$$\cdot\, q(\vec{r}',\vec{\Omega}_1',v_1,t+\frac{s}{v})\cdots q(\vec{r}',\vec{\Omega}_i',v_i',t+\frac{s}{v})d\vec{\Omega}_1'dv_1'\cdots d\vec{\Omega}_i'dv_i'+\cdots\Big]. \quad (1)$$

The integrals over $\vec{r}'$ or $\vec{r}''$ are taken along the direction $(\vec{\Omega})$ of neutron motion. Here the first term is the probability that the neutron escapes from the system without collision; the second is the probability that the neutron has a collision from which no neutrons emerge; the third is the probability of a collision from which one neutron emerges but does not lead to a divergent chain etc.

If we now differentiate q along the direction $\vec{s}$, we obtain an integro-differential equation. Differentiating the right-hand side, we obtain from the exponentials (differentiated with respect to lower limit of integration) the term $+\sigma q$ plus additional terms from the collision integrals. Thus dq/ds is

$$\vec{\Omega}\ \ \text{grad}\ q(\vec{r},\vec{\Omega},v,t)+\frac{1}{v}\frac{\partial q(\vec{r},\vec{\Omega},v,t)}{\partial t}=\sigma(\vec{r},\vec{\Omega},v,t)q(\vec{r},\vec{\Omega},v,t)$$

$$-\sigma(\vec{r},\vec{\Omega},v,t)\Big[c_0(\vec{r},\vec{\Omega},v,t)+\int\int dv'd\vec{\Omega}'c_1(\vec{r},\vec{\Omega},v,t\to\vec{\Omega}',v')q(\vec{r},\vec{\Omega}',v',t)$$

$$+\cdots+\int\int\cdots\int\int dv'd\vec{\Omega}'\cdots dv_i'd\vec{\Omega}_i'c_i(\vec{r},\vec{\Omega},v,t\to\vec{\Omega}',v',\ \cdots,\ \vec{\Omega}_i'v_i')$$

$$\cdot\, q(\vec{r},\vec{\Omega}',v',t)\cdots q(\vec{r},\vec{\Omega}_i'v_i't)+\cdots\Big]. \quad (2)$$

This is our fundamental equation for $q(\vec{r},\vec{\Omega},v,t)$. It is to be solved subject to the boundary conditions that $q(\vec{r}_s,\vec{\Omega},v,t) = 1$ for all $\vec{\Omega} \cdot \vec{n} > 0$, where $\vec{n}$ is an outward normal to the boundary surface at $\vec{r}_s$. (Note that if we deal with a re-entrant system, the boundary surface can be taken well outside of the material boundary.) Evidently a corresponding equation for p can be obtained by substituting $q = 1 - p$ in Eq. (2). The result is

$$\vec{\Omega} \cdot \text{grad } p + \frac{1}{v}\frac{\partial p}{\partial t} = \sigma p - \sigma\left\{\iint dv' d\vec{\Omega}' c_1(\vec{r},\vec{\Omega},v,t \rightarrow \vec{\Omega}',v')p(\vec{r},\vec{\Omega}',v',t)\right.$$

$$+ \ldots + \iint\ldots\iint dv' d\vec{\Omega}' \ldots dv_i' d\vec{\Omega}_i' c_i\left[1 - \left(1 - p(\vec{r},\vec{\Omega}',v',t)\right)\right.$$

$$\left.\left.\ldots \left(1 - p(\vec{r},\vec{\Omega}_i',v_i',t)\right)\right] + \ldots\right\} \qquad (3)$$

with the boundary condition $p(\vec{r}_s) = 0$ if $\vec{\Omega} \cdot \vec{n} > 0$.

In principle, Eq. (3) could be solved directly. However, very little is known about the functions c_i for fission, and considerable simplification results if we assume that each fission neutron emerges isotropically in the $\vec{r}$ system and uncorrelated in energy with other fission neutrons. Then each fission neutron has the probability $f(v')$ of emerging with velocity v', where $f(v')$ is the normalized fission spectrum $\left(\int f(v')dv' = 1\right)$. This assumption does not seem to seriously contradict existing fission theories.[4,5] With this assumption, σc_i may be written $(i > 1)$

$$\sigma c_i = \sigma_f(\vec{r},\vec{\Omega},v,t)c_i^*(\vec{r},\vec{\Omega},v,t)f(v') \ldots f(v_i'), \qquad (4)$$

where σ_f is the fission cross section and c_i^* is the probability of i neutrons emerging from fission. Then the c_i term in Eq. (3) becomes

$$\sigma_f(\vec{r},\vec{\Omega},v,t)c_i^*(\vec{r},\vec{\Omega},v,t)\left[1 - \left(1 - \int dv' f(v')p(\vec{r},v',t)\right)^i\right]$$

$$= \sigma_f c_i^*\left[i \int dv' f(v')p(\vec{r},v',t) - \frac{i(i-1)}{2!}\left(\int dv' f(v')p(\vec{r},v',t)\right)^2\right.$$

$$\left. + \frac{i(i-1)(i-2)}{3!}\left(\int dv' f(v')p(\vec{r},v',t)\right)^3 + \ldots\right]. \qquad (5)$$

Here $p(\vec{r},v,t) = \int d\vec{\Omega} p(\vec{r},\vec{\Omega},v,t)$.

If we now assume that all collisions are either scattering (σ_s), fission (σ_f), or absorption ($\sigma_a = \sigma - \sigma_s - \sigma_f$) and let $\sum_{i=0} ic_i^* = \bar{\nu}(\vec{r},\vec{\Omega},v,t)$ and

$$\sum_{i=0} \frac{i(i-1)\ldots(i-n)}{(n+1)!} c_i^* = \frac{\chi_{n+1}(\vec{r},\vec{\Omega},v,t)}{(n+1)!}, \qquad (6)$$

then Eq. (3) becomes

$$\vec{\Omega} \cdot \text{grad}\, p(\vec{r},\vec{\Omega},v,t) + \frac{1}{v}\frac{\partial p}{\partial t} = \sigma p - \sigma_s \iint dv' d\vec{\Omega}' c_s(\vec{r},\vec{\Omega},v,t \to \vec{\Omega}',v')p(\vec{r},\vec{\Omega}',v',t)$$

$$- \sigma_f(\vec{r},\vec{\Omega},v,t)\left\{\bar{\nu}(\vec{r},\vec{\Omega},v,t)\int f(v')p(\vec{r},v',t)dv' - \frac{\chi_2(\vec{r},\vec{\Omega},v,t)}{2!}\right.$$

$$\left.\left[\int f(v')p(\vec{r},v',t)dv'\right]^2 + \frac{\chi_3(\vec{r},\vec{\Omega},v,t)}{3!}\left[\int f(v')p(\vec{r},v',t)dv'\right]^3 - \ldots\right\}, \qquad (7)$$

where c_s is c_1 for scattering (elastic or inelastic). If one wishes to allow for (n,2n) or (n,3n) reactions, the appropriate additional terms can be easily found from Eq. (3).

It is interesting to note that if the nonlinear terms were removed from Eq. (7), one would obtain an equation in which p could be interpreted as a time-dependent importance function.[6] In fact, p could be the expected number of neutrons in the system at some late time, t_f, due to one neutron present at time t. To see this, let us write down the integral equation for $\bar{n}(\vec{r},\vec{\Omega},v,t;t_f)$, the average number of neutrons in the system at t_f due to one at t (at $\vec{r}$ with velocity $v\vec{\Omega}$). As in Eq. (1),

$$\bar{n}(\vec{r},\vec{\Omega},v,t;t_f) = \int_{\vec{r}}^{\vec{r}_s} d\vec{r}'\,\sigma(\vec{r}',\vec{\Omega},v,t+\tfrac{s}{v})\,\exp\left(-\int_{\vec{r}}^{\vec{r}'}\sigma(\vec{r}'',\vec{\Omega},v,t+\tfrac{s'}{v})\,d\vec{r}''\right)$$

$$\cdot\left\{\iint d\vec{\Omega}'dv'\,c_1(\vec{r}',\vec{\Omega},v,t+\tfrac{s}{v}\rightarrow\vec{\Omega}',v')\bar{n}(\vec{r}',\vec{\Omega}',v',t+\tfrac{s}{v};t_f)\right.$$

$$+\iint d\vec{\Omega}_1'dv_1'd\vec{\Omega}_2'dv_2\,c_2(\vec{r}',\vec{\Omega},v,t+\tfrac{s}{v}\rightarrow\vec{\Omega}_1',v_1',\vec{\Omega}_2',v_2')$$

$$\left.\cdot\left[\bar{n}(\vec{r}',\vec{\Omega}_1',v_1',t+\tfrac{s}{v};t_f)+\bar{n}(\vec{r}',\vec{\Omega}_2',v_2',t+\tfrac{s}{v};t_f\right]+\cdots\right\}. \qquad (8)$$

If we differentiate along the direction $\vec{\Omega}$ and assume uncorrelated fission neutrons, we obtain Eq. (7) with $\bar{n}$ replacing p and all nonlinear terms missing (thus formally $\chi_2 = \chi_3 = \chi_i = 0$). Evidently the initial condition on $\bar{n}$ is $\bar{n} = 1$ at $t = t_f$, and the boundary condition is $\bar{n}(\vec{r}_s, \ldots) = 0$ if $\vec{\Omega}\cdot\vec{n} > 0$. Thus $\bar{n}$ has a different initial condition than p but the same boundary condition.

Note that for large $t_f - t$, $\bar{n}$ will be composed only of neutrons from divergent chains. Thus the ratio $\bar{n}/p$ is the average number of neutrons at time t_f in a divergent chain, which can thus be found as a function of $\vec{r}$, $\vec{\Omega}$, v, and t.

Let us now consider the form of Eq. (7) for a stationary system.

For such a case, $\partial p/\partial t = 0$, and the cross sections have no $\vec{\Omega}$ dependence, although c_s will be a function of $(\vec{\Omega} \cdot \vec{\Omega}')$. We then have

$$\vec{\Omega} \cdot \text{grad } p(\vec{r},\vec{\Omega},v) = \sigma p - \sigma_s \iint dv' d\vec{\Omega}' c_s(\vec{r},\vec{\Omega},v \rightarrow \vec{\Omega}',v') p(\vec{r},\vec{\Omega}',v')$$

$$- \sigma_f(\vec{r},v)\left[\bar{\nu}(\vec{r},v)\left(fp(\vec{r})\right) - \frac{\chi_2(\vec{r},v)}{2!}\left(fp(\vec{r})\right)^2 + \frac{\chi_3(\vec{r},v)}{3!}\left(fp(\vec{r})\right)^3 - \ldots\right], \quad (9)$$

where

$$fp(\vec{r}) = \int f(v')p(\vec{r},v')dv', \quad (10)$$

with the boundary condition $p(\vec{r}_s,\vec{\Omega},v) = 0$ if $\vec{\Omega} \cdot \vec{n} > 0$.

Except for the terms involving σ_f, this equation (plus boundary condition) is identical to that for the neutron adjoint for eigenvalue k. The equation for the adjoint has the fission term,

$$\frac{\sigma_f(\vec{r},v)\bar{\nu}(\vec{r},v)}{k}\left(fp(\vec{r})\right)$$

instead of

$$\sigma_f(\vec{r},v)\left[\quad\right],$$

where k is the eigenvalue. It follows that any DSN or other technique for computing a neutron adjoint can formally be made applicable to a probability calculation by introducing the nonlinear terms of Eq. (9) in place of $\left(\frac{1}{k} - 1\right)\sigma_f\bar{\nu}\left(fp(\vec{r})\right)$.

THE SNP CODE

The SNP code is a straightforward modification of the new DSN code* which then computes probabilities for stationary systems and uncorrelated fission neutrons; that is, it solves Eq. (9), subject to rigorous conservation of $p(\vec{r},\vec{\Omega},t)$, using equations of reference 9.

Previous experience with one-velocity calculations had shown that straightforward iteration of Eq. (9) was a very slowly convergent process for small p. Therefore the SNP code starts by computing the adjoint k distribution (nonlinear terms neglected), then multiplies this distribution by a normalization factor to satisfy the integral of Eq. (9) over the whole system. It then proceeds to iterate Eq. (9) but after each outer iteration, it scales the solution again. This procedure appears to converge very rapidly, and the time for a p calculation is not much longer than for an adjoint k calculation. The procedure is now described in more detail.

In multigroup form and for spherical geometry and isotropic scattering, Eq. (9) can be written as

$$-\vec{\Omega} \cdot \text{grad}\, p_g(r,\mu) + \sigma_g(r)p_g(r,\mu) = S_g(r), \qquad g = 1,2, \ldots G \qquad \text{(11a)}$$

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*The new DSN code uses slightly different difference equations from the DSN code discussed in references 7 and 8 and enforces rigorous neutron conservation.[9]

where

$$S_g(r) = \sum_{g'=1}^{G} \sigma_{s,g\to g'}(r)p_{g'}(r) + \sigma_{fg}(r)\nu_g(r)P(r)$$
$$- \sigma_{fg}(r) \sum_{i=2}^{5} (-1)^i \frac{\chi_i(r)}{i!} P^i(r)$$
$$= \sum_{g'=1}^{G} \sigma_{s,g\to g'}(r)p_{g'}(r) + Q_g(r) \qquad (11b)$$

and

$$p_g(r) = \frac{1}{2}\int_{-1}^{1} p_g(r,\mu)d\mu, \qquad P(r) = \sum_{g'=1}^{G} f_{g'}p_{g'}(r), \qquad (11c)$$

with the boundary condition $p_g(r_{max},\mu) = 0$, $\mu \geq 0$. We assume that not more than five neutrons are emitted in fission. Although the above equations are written for spherical geometry and isotropic scattering, the SNP code can also be applied to planes and infinite cylinders, and a first order anisotropic scattering approximation can be used. Evidently the same procedures can be used for multidimensional geometries.

The calculation begins by solving the linear eigenvalue problem in which $S_g(r)$ is replaced by $S_g^0(r)$ where

$$S_g^0(r) = \sum_{g'=1}^{G} \sigma_{s,g\to g'}(r)p_{g'}(r) + \frac{\sigma_{fg}(r)\nu_g(r)P(r)}{k}. \qquad (12)$$

Once this solution has been found in the usual DSN manner, we test to make sure that $k > 1$. If so, we then multiply $p_g(r)$ by the constant factor λ, where λ is chosen so that

$$\sum_g \int d\vec{r}\, \sigma_{fg}(r)\nu_g(r)P(r)\left(\frac{1}{k} - 1\right) = -\sum_g \int d\vec{r}\sigma_{fg}(r) \sum_{i=2}^{5}(-1)^i \cdot \frac{\chi_i(r)}{i!} P^i(r)\lambda^{i-1}. \quad (13)$$

$\lambda P(r)$ is then used as the first guess for the solution to Eq. (11). The purpose of this normalization is to insure that if $\lambda P(r)$ is used for computing $Q(r)$, then the integral of Eq. (11a), over angle and the volume of the system, is satisfied. It is analogous to the scaling procedure[8] which has been found useful in ordinary DSN calculations. $Q_g(r)$ is formed from $\lambda P(r)$ and used as a fixed source in Eq. (11b) during the next iteration. At the end of the next iteration, we have found a new $p_g(r)$ and new $P(r)$, and we compute a new value of k as the ratio of new $\Sigma \int \sigma_f \nu P$ to old $\Sigma \int \sigma_f \nu P$. A new scale factor is determined such that the integral of Eq. (11a) over the whole system holds when <u>new</u> values of $p_g(r)$ are used throughout. Thus λ is found from

$$\sum_g \int d\vec{r}\sigma_{fg}(r)\nu_g(r)P(r)\left(\frac{1}{k} - 1\right) + \sum_g \int Q_g(r)d\vec{r} = -\sum_g \int d\vec{r}\sigma_{fg}(r) \sum_{i=2}^{5}(-1)^i \frac{\chi_i(r)}{i!} P^i(r)\lambda^{i-1}, \quad (14)$$

and this new value of $\lambda P_g(r)$ is used to compute $Q_g(r)$ which is held constant through the next iteration. This procedure is continued until both k and λ are sufficiently close to unity. In practice, the first value of λ (obtained from Eq. (13)) is usually very far from unity, being typically $\sim 10^3$, and to avoid difficulty in finding the correct root of Eq. (13), we

have ignored the P^3 and higher terms in obtaining this first λ. Subsequent values of λ are quite close to unity and for them, the quartic equation for λ, Eq. (14), is solved. Convergence has been found to be quite rapid, requiring only eight or so iterations on λ to give p to ~.1%.

Data input for a SNP problem is the same as for a DSN problem except that the eigenvalue index is set to -1 (I04 = -1 in new DSN), and the additional parameters $\sigma_f\chi_2/2!$, $\sigma_f\chi_3/3!$, $\sigma_f\chi_4/4!$, and $\sigma_f\chi_5/5!$ are entered in front of the absorption cross section in the cross section table for each element. These parameters are discussed in the following section.

ADDITIONAL FISSION PARAMETERS

Terrell[5] studied the probability of obtaining various numbers of neutrons from fission. He found that all the experimental data on neutron multiplicity could be fit by a universal distribution. If c_n is the probability of obtaining n neutrons, he found

$$\sum_{n=0}^{\nu} c_n = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{(\nu-\bar{\nu}+1/2)/\sigma} e^{-t^2/2} dt,$$

with $\sigma = 1.08$. With this distribution, c_n is only a function of $\bar{\nu}$. We have used the distribution to compute $\sum_{n=m}^{\infty} n(n-1) \ldots (n-m+1)c_n$. The calculations were performed by Mrs. J. Powers, and in Figure 1 are plotted $\chi_2/2!\bar{\nu}$, $\chi_3/3!\bar{\nu}$, $\chi_4/4!\bar{\nu}$, and $\chi_5/5!\bar{\nu}$ vs. $\bar{\nu}$. For use in the SNP calculation, these numbers are multiplied by $\bar{\nu}\sigma_f$ and entered as the first four entries of each cross section table (they are entered for each material and group). Macroscopic cross sections are formed from these microscopic cross sections by the code in the usual way.

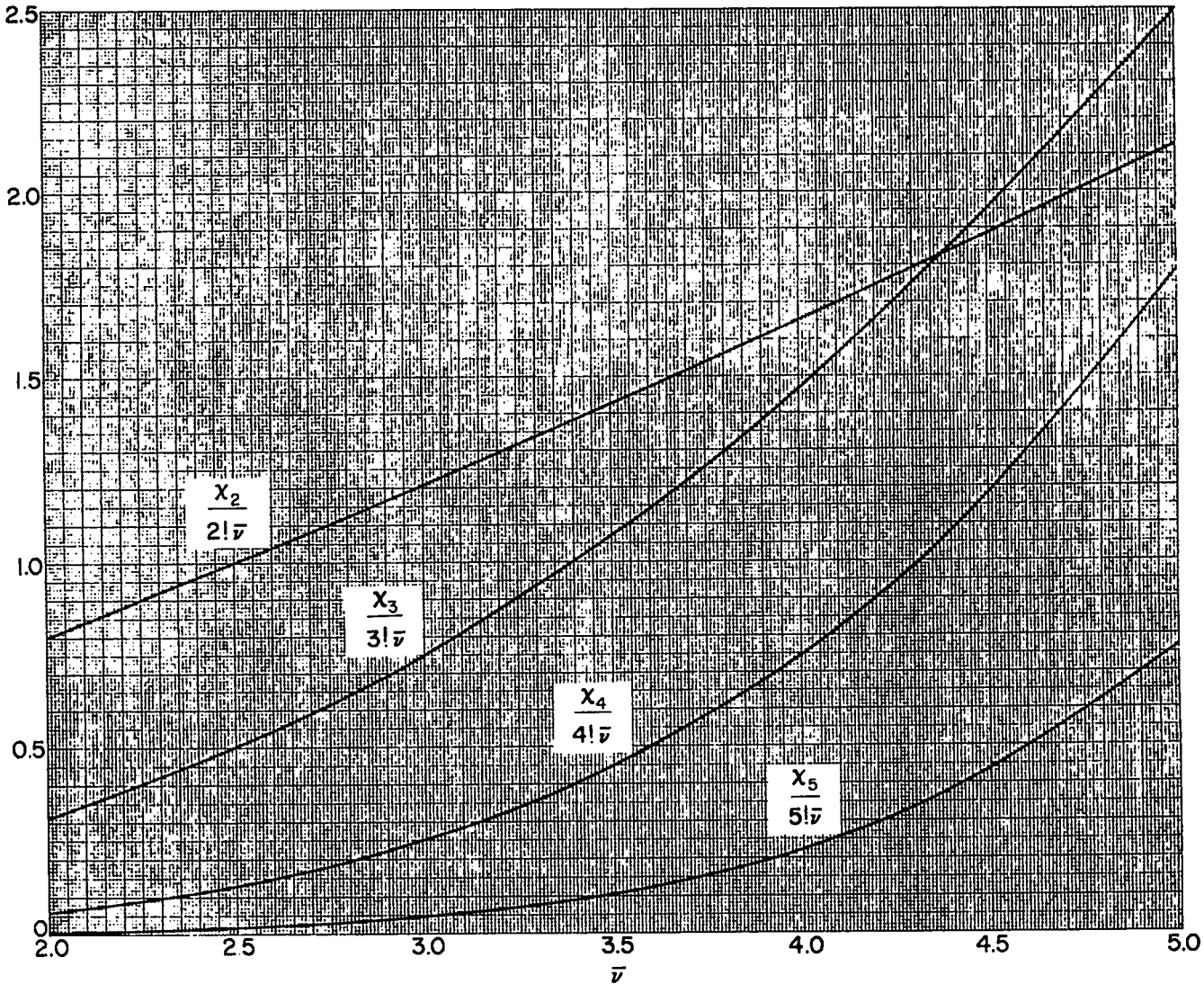

Figure 1 Fission parameters as functions of $\bar{\nu}$

TEST PROBLEMS

A number of problems were run with the code by R. Anderson for comparison with previous results. First of all, a very slightly supercritical system was run for comparison with the linear problem. For simplicity in computing integrals, the system was a one group uniform

A theory for the very slightly supercritical case was sketched by Feynman for the integral equation approach. We here show, for the one group case, how the theory can be applied here. Generalization to many groups is straightforward. For a slightly supercritical system, $p(r)$ will be small, and in Eq. (11) we ignore terms in p^3 and higher powers of p. Equation (11) can then be written

$$-\vec{\Omega} \cdot \text{grad } p(r,\mu) = (c - 1)p(r) - \frac{\chi_2}{2} p^2(r), \tag{15a}$$

where we have set $\sigma = 1$, c is the average number of secondary neutrons per collision, and $\chi_2/2$ the mean number of pairs. We consider the corresponding linear eigenvalue equations for the neutron flux $n(r)$ and adjoint $n^+(r)$:

$$-\vec{\Omega} \cdot \text{grad } n_i^+(r,\mu) = (c_i - 1)n_i^+(r), \tag{15b}$$

$$\vec{\Omega} \cdot \text{grad } n_i(r,\mu) = (c_i - 1)n_i(r), \tag{15c}$$

where n_i is the eigenfunction corresponding to the eigenvalue c_i. Assuming, as is customary, that the functions $n_i^+(r,\mu)$ form a complete set, we expand

$$p(r,\mu) = \sum_{i=0}^{\infty} a_i n_i^+(r,\mu). \tag{16}$$

Substituting in Eq. (15a), we have

$$\sum_{i=0}^{\infty} a_i(c_i - c)n_i^+(r) = -\frac{\chi_2}{2}\left(\sum_{i=0}^{\infty} a_i n_i^+(r)\right)^2. \tag{17}$$

Let us now multiply this by $n_j(r)$ and integrate over the volume. Since $n_j(r)$ and $n_i^+(r)$ are orthogonal, we have

$$a_j(c_j - c)\int n_j^+(r)n_j(r)r^2dr = -\frac{\chi_2}{2}\int r^2drn_j(r)\left(\sum_{i=0}^{\infty} a_i n_i^+(r)\right)^2. \qquad (18)$$

Now in the p^2 term within the integral, one expects that the term $\left(a_0 n_0^+(r)\right)^2$ is the largest. If so, we may approximate Eq. (18) by

$$a_j(c_j - c)\int n_j^+(r)n_j(r)r^2dr \simeq -\frac{\chi_2}{2}\int a_0^2\left(n_0^+(r)\right)^2 n_j(r)r^2dr, \qquad (19)$$

which for $j = 0$ gives

$$a_0 n_0^+ \simeq \frac{(c - c_0)}{\chi_2/2}\frac{\int n_0^+(r')n_0(r')r'^2dr'}{\int\left(n_0^+(r')\right)^2 n_0(r')r'^2dr'}\, n_0^+(r), \qquad (20a)$$

while for $j \neq 0$,

$$a_j n_j^+(r) = -\frac{\chi_2}{2(c_j - c)}\frac{\int a_0^2\left(n_0(r')\right)^2 n_j(r')r'^2dr'}{\int n_j^+(r')n_j(r')r'^2dr'}\, n_j^+(r). \qquad (20b)$$

From Eq. (20a), we see that $a_0 n_0^+$ is of order $(c - c_0)$ which is a very small number for a slightly supercritical system. But Eq. (20b) then

tells us that $a_j n_j^+$ is of order

$$\frac{\chi_2}{2(c_j - c)} (c - c_0)^2 \sim (c - c_0)^2,$$

which is the square of a very small number. This confirms our assumption that $a_0 n_0^+(r)$ is a good approximation to $p(r)$, and we conclude that $p(r)$ will be given by Eq. (20a) for very small $c - c_0$.

Note that Eq. (20a) is essentially a determination of a normalization constant or scale factor a_0. It differs from Eq. (13), which we use for finding a scale factor first of all trivially, in that Eq. (13) is in multigroup form and retains p^3 and higher terms but also essentially in that in Eq. (20a) both numerator and denominator have been multiplied by the weighting function $n_0(r)$. This makes Eq. (20a) more accurate for slightly supercritical systems, but if we wished to employ such a weighting in practice, this would require an additional calculation of $n_0(r)$ and very likely lengthen the calculation. Of course, for a one group calculation, $n_0(r) = n_0^+(r)$, but in general this is not the case.

A one group test problem (a 9.5 cm Oy sphere) was run to see if the SNP code calculates $p(r)$ in agreement with Eq. (20a). Good agreement was found.

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